Numerical Solutions of Euler Equations Using Simplified Flux Vector Splitting

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Abstract

A NUMERICAL method for solving the two-dimensional Euler equations for steady-state solutions using flux vector splitting is developed. The equations are expressed in curvilinear coordinates and the finite volume approach is used. The energy equation is omitted since only steady-state solutions are required. A simplified implicit operator is employed to reduce the computational effort of the present method. Convergence characteristics are compared with predictions obtained by other authors.

Contents

In the present research effort, the work of von Lavante and Trevino[1] is further extended to provide a simple, efficient method for solving the Euler equations. Under the assumption of constant stagnation enthalpy, the two-dimensional Euler equations in vector form for general, body-fitted coordinates written in nondimensional strong conservation law form are

$$\frac{\partial Q}{\partial t}+\frac{\partial F}{\partial \xi}+\frac{\partial G}{\partial \eta}=0 \tag{1}$$

where

$$Q=\frac{1}{J}\begin{bmatrix}\rho\\ \rho u\\ \rho v\end{bmatrix}\quad F=\frac{1}{J}\begin{bmatrix}\rho U_\xi\\ \rho u U_\xi+p\xi_x\\ \rho v U_\xi+p\xi_y\end{bmatrix}\quad G=\frac{1}{J}\begin{bmatrix}\rho U_\eta\\ \rho u U_\eta+p\eta_x\\ \rho v U_\eta+p\eta_y\end{bmatrix}$$

with the equation of state

$$p=\frac{\rho}{\gamma}\cdot\left\{1-\frac{\gamma-1}{2}\cdot(u^2+v^2)\right\} \tag{2}$$

J is the Jacobian of the transformation of coordinates, ρ is density, u and v velocity components, p is static pressure, and U_ξ and U_η are the contravariant velocities. An implicit Euler single-step temporal scheme was selected for advancing the solution of Eq. (1) in time. After linearization in time of the flux vectors F and G, and approximate factorization of the implicit operator (see Ref. 1 for more details), the basic algorithm has the form

$$[I+\Delta t\delta_\xi A^n]\,[I+\Delta t\delta_\eta B^n]\,\Delta Q^n=-\Delta t(\delta_\xi F^n+\delta_\eta G^n)=R^n$$
$$Q^{n+1}=Q^n+\Delta Q^n \tag{3}$$

where A and B are the Jacobian matrices $A=\partial F/\partial Q$, $B=\partial G/\partial Q$. A and B each have a complete set of eigenvalues

$$\lambda_1=U_\xi,\ \lambda_2=U_\xi(\gamma+1)/2\gamma+SQ,\ \lambda_3=U_\xi(\gamma+1)/2\gamma-SQ \tag{4}$$

with $SQ=\sqrt{[U_\xi(\gamma-1)/2\gamma]^2+c^2(\xi_x^2+\xi_y^2)/\gamma}$ and $c=$ speed of sound. The eigenvalues of B are similar and are obtained by replacing ξ_x, ξ_y, by η_x, η_y. The Jacobian matrix A can be split into three matrices each corresponding to only one eigenvalue. Then, using the homogeneous property of the flux vectors F and G results in

$$F=A\cdot Q=A^1Q+A^2Q+A^3Q=F^1+F^2+F^3 \tag{5}$$

The resulting split flux parts of the vector F are given in the full version of this article. It should be noted that this type of splitting is effectively the same as the Steger-Warming splitting.[2] The split flux vector $F^1=0$. After substituting the split flux vectors F^2 and F^3 into Eq. (3), the solution scheme is given by

$$\left[I+\Delta t\left(\frac{\partial}{\partial\xi}\bar{A}^2+\frac{\partial}{\partial\xi}\bar{A}^3\right)\right]^n \times\left[I+\Delta t\left(\frac{\partial}{\partial\eta}\bar{B}^2+\frac{\partial}{\partial\eta}\bar{B}^3\right)\right]^n\cdot\Delta Q^n =-\Delta t\cdot\left[\frac{\partial F^2}{\partial\xi}+\frac{\partial F^3}{\partial\xi}+\frac{\partial G^2}{\partial\eta}+\frac{\partial G^3}{\partial\eta}\right]^n=-\Delta t\cdot R^n \tag{6}$$

where, for example, $\bar{A}^2=\partial F^2/\partial Q$.

Finite-volume-type discretization was chosen because of its general applicability and ability to preserve the conservation properties of the governing equations. The computational domain is subdivided into a set of nonoverlapping cells (control volumes), formed by lines that connect the intersection points of diagonal lines between the grid points. The explicit part of Eq. (3), written for one such control volume, can be converted from a surface integral into a closed curve integral. Assuming that $q_{i,j}$ represents average flow quantities in the cell surrounding the grid point i,j and F,G are constant on the corresponding sides of the cell, Eq. (3) can be written as

$$-\Delta tR^n=-(\Delta t/S)\,[F^2_{i+\frac12,j}-F^2_{i-\frac12,j}+F^3_{i+\frac12,j}-F^3_{i-\frac12,j} +G^2_{i,j+\frac12}-G^2_{i,j-\frac12}+G^3_{i,j+\frac12}-G^3_{i,j-\frac12}] \tag{7}$$

$F^3_{i+\frac12,j}$ denotes, for example, the value of flux F^3 on the cell boundary between the grid points i,j and $i+1,j$, and S is the area of the cell.

The fluxes were evaluated at cell interfaces using the MUSCL-type finite volume formulation in a way similar to that in Ref. 3. Here, the values of q, from which the fluxes are formed, are extrapolated from one side, depending on

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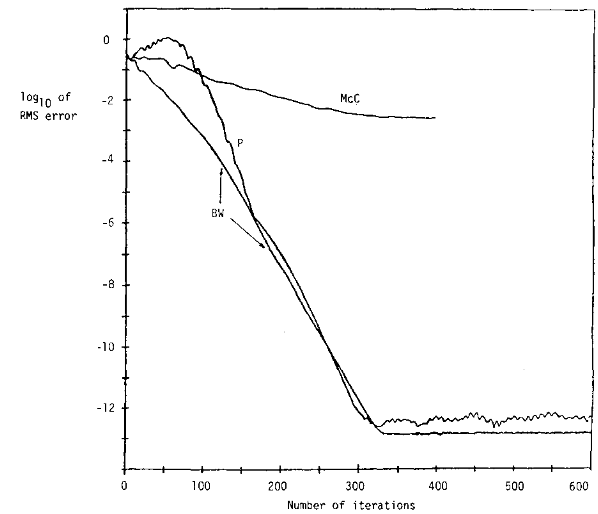

Fig. 1 Residual history for supersonic diffusor: P, present scheme, CFL = 2.4; McC, implicit MacCormack scheme, CFL = 3.5; BW, Beam-Warming scheme, CFL = 4.0.

the sign of the corresponding eigenvalue. Two-point extrapolation was chosen for second-order accuracy in space. In the case where λ changes its sign between two points, i.e., $\lambda_{i-1}^3 < 0$ and $\lambda_i^3 > 0$, the values of $q_{i-\frac{1}{2}}$ are interpolated between i and $i-1$ using a weighted average.

The most logical form of the implicit operator is given by Eq. (6). However, the determination of the full Jacobian matrices $\bar{A}^2$, $\bar{A}^3$, $\bar{B}^2$, and $\bar{B}^3$ is computationally involved, and the inversion of the resulting block tridiagonal systems of equations is time-consuming. Since only steady-state results were of interest, the implicit part can be simplified. This typically reduces the stability limits, but the net effect might be reduced computational time due to its simplicity. Therefore, a simple operator, applied already by von Lavante and Trevino,[1] was chosen.

Here, following the procedure in Ref. 2, the Jacobian matrices in the implicit part of Eq. (3) are first split into nonnegative and nonpositive parts, and then substituted by their spectral radii evaluated at the center of the cell i,j.

Using for example, $\lambda_{Sr,A}^+ = \max(|\lambda_{A,i}^+|)$, Eq. (3) finally becomes

$$[1+\Delta t\delta_\xi^b(\lambda_{Sr,A}^+)^n - \Delta t\delta_\xi^f(\lambda_{Sr,A}^-)^n][1+\Delta t\delta_\eta^b(\lambda_{Sr,B}^+)^n - \Delta t\delta_\eta^f(\lambda_{Sr,B}^-)^n]\Delta Q^n = -\Delta t[\delta_\xi F^2 + \delta_\xi F^3 + \delta_\eta G^2 + \delta_\eta G^3] \tag{8}$$

The implicit part of Eq. (8) uses first-order upwind differencing; the accuracy of the explicit part depends on the type of extrapolation of the vector Q to cell faces. This means that the above scheme will, despite all of the simplifications, still give second-order-accurate steady-state results. The resulting algorithm [Eq. (8)] yields a scalar tridiagonal set of equations that can be efficiently solved.

Purely explicit boundary conditions were used due to their simplicity and general applicability. They were very similar to those used in Ref. 1 and will not be repeated here.

Results

The performance of the present algorithm was tested on several two-dimensional internal flow problems, which are discussed in the full version of this paper. The results obtained for computations of flows in a supersonic diffusor and transonic compressor cascade will be discussed here since they represent typical test cases.

Supersonic Diffusor

The flow in a supersonic diffusor with a weak oblique shock generated by a 3-deg compression corner with the inflow Mach number of 2.0 was predicted using a numerically generated 51×15 grid. The residual history can be seen in Fig. 1. The best rate of convergence was achieved at the maximum stable CFL number of 2.4. The rate of convergence for the present method was compared with the convergence properties of two methods, the implicit MacCormack scheme[4] and the basic form of the Beam-Warming scheme.[5] The MacCormack method had maximum rate of convergence at a CFL number of 3.5 at which the residual decreased approximately three orders of magnitude and then remained at a relatively high level. The Beam-Warming scheme had a rate of convergence similar to the present method. The Beam-Warming method converged after approximately the same number of iterations, but due to its higher CPU time per iteration, this required more than three times more CPU time. It should be pointed out that even the faster diagonal version of the Beam-Warming scheme required about 60% more computational time than the present method.

Transonic Compressor Cascade

The performance of the present method was further tested on a generic transonic compressor cascade used already by Bush.[6] The inflow Mach number was 1.38, the inflow angle was 64.14 deg, and the static pressure ratio across the cascade was 2.09. The blade surface pressure results agreed reasonably well with results obtained by Bush.[6] The relatively strong shocks ($M = 2$ before the shock) were smeared only over two to three points. The rms error decreased four orders of magnitude after 1000 iterations at a maximum stable CFL number of 0.7. Although this CFL number is relatively low, the efficiency of the method is still, mainly due to its simplicity, competitive with other comparable methods.[1,4-6]

Conclusion

A simplified form of the flux vector splitting algorithm in finite-volume formulation was introduced. The performance of the resulting algorithm was tested on several two-dimensional internal geometries. The results show that the present method is, due to its simplicity, computationally efficient. It does not require the addition of artificial damping terms due to its natural dissipation, and thus eliminates one user-selectable parameter. Its low maximum stable CFL number for low Mach number cases is due to the simple implicit operator and needs further study.

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